Combinatorics in Banach space theory (MIM UW 2014/15)

PROBLEMS (Part 1)

PROBLEM 1.1. Give an example of a coloring $[\mathbb{N}]^\infty \to \{-1,1\}$ which is both Lebesgue and Baire measurable, when regarding $[\mathbb{N}]^\infty$ as a subset of the Cantor set $\{0,1\}^\mathbb{N}$, but is not constant on any set of the form $[M]^\infty$, for any infinite $M \subset \mathbb{N}$.

PROBLEM 1.2. Show that every completely Ramsey set $\mathcal{V} \subset [\mathbb{N}]^\infty$ has the Baire property with respect to the Ellentuck topology.

Hint. This may be proved, e.g., by showing that $\mathcal{V} \setminus \mathrm{int}\mathcal{V}$ is nowhere dense.

PROBLEM 1.3. Show that every set $\mathcal{V} \subset [\mathbb{N}]^\infty$ that is meager in the Ellentuck topology must be nowhere dense. In fact, show that for every basic open set $[a, A]$ there exists $B \in [a, A]$ such that $[a, B] \cap \mathcal{V} = \varnothing$. (Recall the notation

$$[a, A] = \{C \in \mathcal{P}_\infty\mathbb{N} \colon a \subset C \subseteq a \cup A,\ a < C \setminus a\}$$

for $a \in [\mathbb{N}]^{<\infty}$ and $A \in [\mathbb{N}]^\infty$.)

Remark. This is the key observation in order to prove that every subset of $[\mathbb{N}]^\infty$ with the Baire property with respect to the Ellentuck topology is completely Ramsey, knowing already that all Ellentuck-open sets are completely Ramsey.

PROBLEM 1.4. The *Schreier space* $\mathcal{S}$ is the completion of $(c_{00}, \|\cdot\|)$ under the norm given by

$$\|x\| = \sup\left\{\sum_{n\in E} |x_n| \colon E \subset \mathbb{N} \text{ and } |E| \leqslant \min E\right\}.$$

Verify that the spreading model of the canonical basis of $\mathcal{S}$ is isomorphic to ℓ_1.

PROBLEM 1.5. Let $(x_n)_{n=1}^\infty$ be a *spreading* basic sequence in a Banach space, that is, for any integers $0 < p_1 < \ldots < p_n$ and any scalars $(a_j)_{j=1}^n$ we have

$$\left\|\sum_{j=1}^n a_j x_{p_j}\right\| = \left\|\sum_{j=1}^n a_j x_j\right\|.$$

Show that the basic sequence $(x_{2n} - x_{2n+1})_{n=1}^\infty$ is 1-unconditional.

Hint. It is enough to prove that the vectors $y_n := x_{2n} - x_{2n+1}$ satisfy $\|\sum_{i\in I} a_i y_i\| \leqslant \|\sum_{j\in J} a_j y_j\|$ for every pair of finite sets of natural numbers $I \subseteq J$ (why?).

PROBLEM 1.6. Let $(x_n)_{n=1}^\infty$ be a non-constant, weakly null, spreading sequence. Show that it is an unconditional basic sequence with the *suppression constant* $K_s = 1$; note that K_s is defined as the supremum of norms of projections P_A corresponding to all subsets $A \subset \mathbb{N}$ (for $x = \sum_{n=1}^\infty a_n x_n$, $P_A x = \sum_{n\in A} a_n x_n$).

Hint. It is enough to show that for every finite sequence of scalars $(a_j)_{j=1}^k$ and each $j_0 \in \{1, \ldots, k\}$ we have $\|\sum_{j\neq j_0} a_j x_j\| \leqslant \|\sum_j a_j x_j\|$ (why?).

Remark. In general, the suppression and unconditional constants (K_s and K_u, respectively) satisfy $1 \leqslant K_s \leqslant K_u \leqslant 2K_s$, hence every non-constant, weakly null, spreading sequence forms an unconditional basic sequence.

PROBLEM 1.7. Prove one of several James' criterions of (non-)reflexivity: If X is a non-reflexive Banach space, then for every $\theta \in (0,1)$ there exist sequences $(x_n) \subset B_X$ and $(x_n^*) \subset B_{X^*}$ such that $x_n^* x_j = \theta$ for all $n \leqslant j$ and $x_n^* x_j = 0$ for all $n > j$.

Hint. Use Helly's theorem which says that given any functionals $x_1^*, \ldots, x_n^* \in X^*$ and scalars $\alpha_1, \ldots, \alpha_n$, the following assertions are equivalent:

(i) there exists $x \in X$ such that $x_j^* x = \alpha_j$ for $1 \leqslant j \leqslant n$;

(ii) there is a constant γ such that $\left|\sum_{j=1}^n \alpha_j \beta_j\right| \leqslant \gamma \left\|\sum_{j=1}^n \beta_j x_j^*\right\|$ for all scalars $\beta_1, \ldots, \beta_n$.

Moreover, if (ii) holds true, then for every $\varepsilon > 0$ the vector $x \in X$ in assertion (i) may be chosen so that $\|x\| \leqslant \gamma + \varepsilon$.

PROBLEM 1.8. Prove that the unit ball of any non-reflexive, infinite-dimensional Banach space contains an infinite sequence $(x_n)_{n=1}^\infty$ such that

$$\|x_m - x_n\| \geqslant \sqrt[5]{4} \quad \text{for all } m \neq n.$$

Hint. Apply the James criterion of non-reflexivity given in Problem 1.7 and combine it with the Brunel–Sucheston theorem. Consider four types of combinations whose coefficients are given by:

- $a_1^{(1)} = 1$, $a_2^{(1)} = -1$;
- for $i = 2, 3, 4$: $a_j^{(i)} = (-1)^i$ if $1 \leqslant j < i$ or $j = 2i$, and $a_j^{(i)} = (-1)^{i+1}$ if $i \leqslant j < 2i$.

Remark. This gives a concrete estimate for the Elton–Odell theorem in the non-reflexive case. It was proved by A. Kryczka and S. Prus (2000).

PROBLEM 1.9. Prove the following statement called *root lemma* or Δ-*system lemma*: If $\mathcal{A}$ is an uncountable family of finite sets, then there exists an uncountable subfamily $\mathcal{B}$ of $\mathcal{A}$ and a finite (possibly empty) set S such that $A \cap B = S$ for all $A, B \in \mathcal{B}$ with $A \neq B$.

Hint. With no loss generality we may assume that $|\mathcal{A}| = \aleph_1$ and all members of $\mathcal{A}$ are finite subsets of the ordinal interval $[0, \omega_1]$. Show that for some $n \in \mathbb{N}$ the collection $\mathcal{A}_n := \{A \in \mathcal{A} \colon |A| = n\}$ is uncountable and $\sup(\bigcup_{A \in \mathcal{A}_n} A) = \omega_1$. For each $A \in \mathcal{A}_n$ write $A = \{A(1), \ldots, A(n)\}$ with $A(1) < \ldots < A(n)$ and define $p \in \{1, \ldots, n\}$ to be the least integer satisfying $\sup\{A(p) \colon A \in \mathcal{A}_n\} = \omega_1$.

PROBLEM 1.10. Let $(x_\alpha)_{\alpha \in A}$ be any sequence in the unit ball of $c_0(\omega_1)$ satisfying

$$\|x_\alpha - x_\beta\| \geqslant 1 + \varepsilon \quad \text{for all } \alpha, \beta \in A,\ \alpha \neq \beta,$$

where ε is some fixed positive number. Show that A must be countable.

Hint. Apply the root lemma (cf. Problem 1.9).

Remark. This shows that the uncountable version of the Elton–Odell theorem fails to hold true in general for non-separable Banach spaces.

PROBLEM 1.11. Prove *James' c_0-distortion theorem*: Let X be a Banach space containing an isomorphic copy of c_0, and let $(x_n)_{n=1}^\infty \subset X$ be a normalized basic sequence equivalent to the canonical basis of c_0. Then, for every $\delta > 0$ there exists a normalized block basic sequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that

$$(1-\delta) \sup_{1 \leqslant j \leqslant n} |a_j| \leqslant \left\| \sum_{j=1}^n a_j y_j \right\| \leqslant \sup_{1 \leqslant j \leqslant n} |a_j|$$

for any sequence of scalars $(a_j)_{j=1}^n$.

Remark. This statement is completely analogous to James' ℓ_1-distortion theorem. The fact that nothing similar can be said for ℓ_p-spaces with $1 < p < \infty$ (in other words, that they are all *arbitrarily distortable*) is extremely difficult to prove; it was done by Odell and Schlumprecht (1994). The first known example of an arbitrarily distortable space was the Schlumprecht space (1991).

PROBLEM 1.12. Let $(x_n)_{n=1}^\infty$ be any sequence in a Banach space X and let K be any positive constant. Show that the set

$$\mathcal{B}_K := \left\{ M = (m_k)_{k=1}^\infty \in [\mathbb{N}]^\infty \colon \sup_n \left\| \sum_{j=1}^n x_{m_j} \right\| \leqslant K \right\},$$

where $(m_k)_{k=1}^\infty$ stands for the increasing enumeration of M, is closed in the product topology (regarding $[\mathbb{N}]^\infty$ as a subset of the Cantor space $\{0,1\}^{\mathbb{N}}$).

PROBLEM 1.13. Let $(x_n)_{n=1}^\infty$ be a bimonotone basic sequence in a Banach space X, that is, for any convergent series of the form $\sum_{n=1} a_n x_n$, and any $k \in \mathbb{N}$ we have

$$\left\| \sum_{n=1}^k a_n x_n \right\|, \left\| \sum_{n=k+1}^\infty a_n x_n \right\| \leqslant \left\| \sum_{n=1}^\infty a_n x_n \right\|.$$

For any $K > 0$ define $\mathcal{B}_K \subset [\mathbb{N}]^\infty$ as in Problem 1.12. Suppose that a set $M \in [\mathbb{N}]^\infty$ satisfies $[M]^\infty \subseteq \bigcup_{K>0} \mathcal{B}_K$. Prove that for some $M' \in [M]^\infty$ and $K > 0$ we have $[M']^\infty \subseteq \mathcal{B}_K$.

Hint. Use the fact that every pointwise closed set in $[\mathbb{N}]^\infty$ is completely Ramsey. Of course, you should also refer to the assertion of Problem 1.12.

PROBLEM 1.14. Let $(x_n)_{n=1}^\infty$ be a weakly null sequence in a Banach space X and $(x_m^*)_{m=1}^\infty$ be a bounded sequence in X^*. Show that for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|x_m^* x_{n_0}| < \varepsilon$ for infinitely many m's.

Hint. Try first to prove something different. Namely, that for every weakly null sequence $(y_n)_{n=1}^\infty$ and every $\delta > 0$ there exists a finite sequence $(\lambda_j)_{j=1}^k$ of positive numbers summing up to 1 such that

$$\max \left\{ \left\| \sum_{j=1}^k \varepsilon_j \lambda_j y_j \right\| \colon |\varepsilon_j| = 1 \text{ for } 1 \leqslant j \leqslant k \right\} < \delta.$$